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in this course of reasoning we have that $e=k$, or, more generally speaking, that

$$s_\gamma \qquad g<\gamma<g+1$$

either does not interchange any elements of two transitive constituents of G_1 or it interchanges all. Hence the transitive constituents satify the definition of systems.*

We are now prepared to see that the construction of non-primitive groups consists of two operations :

1. The construction of an intransitive group G_1 whose transitive constituents are conjugate groups.†
2. The construction of the substitutions of G which interchange the systems.

The first of these operations is a special case under the construction of intransitive groups and needs therefore no further attention. With respect to the second we shall first consider some special cases and then take up the general problem.

[To be Continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from September-October Number.]

SCHOLION II, *in which is weighed the idea of that brilliant man Giovanni Alfonso Borelli in his Euclides Restitutus.*

This most learned author blames Euclid, because he defines parallel straight lines to be those, *which lying in the same plane do not meet on either side, even if produced into the infinite.*

He offers as ground for his accusation, that such relation is unknown: *first*, he says, *because we are ignorant whether such infinite non-concurrent lines can be found in nature; then also because we cannot perceive the properties of the infinite, and hence a relation of this sort is not clearly cognized.*

But with reverence for so great a man it may be said: can Euclid be blamed, because (to bring forward one among innumerable examples) he defines *a square* to be *a figure quadrilateral, equilateral, rectangular ;* when it may be doubted, whether a figure of this sort has place in nature ? He could, say I, most justly have been blamed, if, before as a Problem demonstrating the construction, he had assumed the aforesaid figure as given.

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*Cf. Netto's Theory of Substitutions (Cole's edition), §67.

†It has not been proved that all groups of this form can be used for G_1 but that every G_1 is of this form. The former cannot be proved.

But that Euclid is free from this fault follows manifestly from this, that he nowhere presumes a square by itself explained, except after Proposition 46 of the First Book, in which in form of a problem he teaches, and demonstrates *the description from a given straight line, of the square as defined by him.*

In the same way therefore Euclid ought not to be blamed, because he defined parallel straight lines in this manner, since he nowhere assumes them as given for the construction of any problem, except after Proposition 31 of book first, in which as a Problem he demonstrates, how *should be drawn from a given point without a given straight line a straight line parallel to this*, and indeed according to the definition of parallels given by him, *so that produced indeed into the infinite on neither side do they meet one another.* And what is more ; he demonstrates this without any dependence from the Postulate here controverted. And so Euclid demonstrates without any petitio principii that *there can be found in nature two such straight lines, which* (lying in the same plane) *protracted on each side into the infinite never meet*, and therefore *makes clearly known to us that relation*, by which he defines parallel straight lines.

Let us continue onward, whither the scrupulous accuser of Euclid invites us. Parallel straight lines he calls any two straights *AC*, *BD*, which stand at right angles to one certain straight *AB* (fig. with me 21). I admit, that a definition of this sort is set forth *by a state* (as he says) possible and most evident ; since (Eu. I, 11.) from any point in the given straight a perpendicular can be erected.

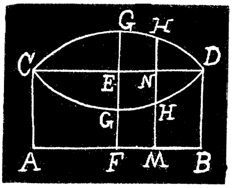

Fig. 21.

But precisely this both possibility and clearness I have just now demonstrated about the definition propounded by Euclid.

Wherefore remains only to compare that known Postulate of Euclid with this the other like postulate, which must be used for farther progress after new definition of parallels.

But behold this other postulate in Clavius (to whom Borelli himself expressly refers) in the Scholion after Proposition 18, book first*:* If a straight line, as suppose *BD* upon another straight, as suppose *BA*, moves transversely making with it at its extremity *B* always right angles, its other extremity *D* describes a line also straight, until this *BD* shall have come to congruence with the other equal *AC*. I acknowledge the fitness of the postulate, that thence a transit may be made to demonstrating that other Euclidean, upon which certainly at length must be supported all remaining geometry. For Clavius had previously declared ; that a line, of which all points are equally distant from a certain assumed straight *AB ;* as assuredly is (from the hypothesis of the aforesaid construction) the line *DC ;* this line also must be straight ; because certainly it will be of this sort, that all its intermediate points *lie ex aequo* (such is the definition of a straight line) between its extreme points *D*, and *C*; *lie ex aequo*, say I, since all are equally distant from this assumed straight *AB*, truly as much as is the length of this *BD*, or *AC*. In this place Clavius introduces the example of the circular line, of which we will speak more conveniently below; where I will show the clearest disparity in this part between the straight line and circle.

For meanwhile I say it is not sufficiently evident, whether the line described by this point *D* is rather the straight *DC* than a certain curve *DGC* either convex or concave toward the side of this *BA*.

For if from the point *F* bisecting this *BA* a perpendicular is supposed erected, which meets the straight *DC* in *E*, and the aforesaid curves in *G*, and *G*, it follows surely (from P. II.) that the angles at the point *E* will be right; whatever line *DC* is understood at length as described in this motion by the point *D*; and moreover (from an easily understood superposition) the angles at the points *G* will be equal according as the one or the other curve *DGC* may be described.

But again; any point *M* in *AB* being assumed; if a perpendicular is erected, which meets the straight *DC* in *N*, and the aforesaid curves in *H* and *H*, I will prove a little later that the angles on both sides at the point *N* will be right, is as far indeed as this straight *DC* is supposed generated by the point *D* in that motion of its, or in as far as the straight *MN* is decided equal to this *BD*.

But if one or the other curve *DHC* is supposed generated; from the like aforesaid easy superposition will be demonstated that again the angles *MHD*, *MHC* on both sides will be equal, wherever in the one or the other described curve the point *H* may be assumed, from which to the straight line *AB* lying under the perpendicular *HM* is understood as let fall. But of this thing more fully and more scrupulously in the other part of this book, where it has its proper place.

To what end therefore, you will say, this untimely anticipation?

To this end, say I; lest from this indubitable property of the line generated in this manner, proved by me most rigorously in the aforesaid place; and indeed beyond any defect of any sort infinitely small; we may decide precipatately that the line can be only the straight.

Obviously the nature of the straight line must here be investigated more profoundly, without which geometry scarcely grown beyond infancy must there remain. Therefore in this affair cannot be blamed a greater investigation of a certain most exact verity.

Nor yet do I here deny, but that by certain most accurate physical experimentation can be discovered, that the line *DC* generated by this motion might be determined not other than a straight line.

But in so far as it may be here allowable to cite physical experimentation, I may forthwith bring forward three demonstrations physico-geometric to sanction the Euclidean Postulate.

Therewith I do not speak of physical experimentation extending into the infinite, and therefore impossible for us; such as of course would be required to the cognizing, that all points of the straight join *DC* are equidistant from the straight *AB*, which is supposed to be in the some plane with this *DC*.

For a single individual case will be sufficient for me; as suppose, if, the straight *DC* being joined, and any one point of it *N* being assumed, the perpendicular *NM* let fall to the underlying *AB* is ascertained to be equal to *BD* or *AC*. For then the angles on both sides at the point *N* would be equal (P. I.) to the angles corresponding to them at the points *C* and *D*, which again (from the same P. I.) would be equal *inter se*. Wherefore the angles on both sides at the point *N*, and therefore also the remaining two will be right.

Therefore we will have a case for the hypothesis of right angle; and consequently (by PP. V. and XIII.) we will have demonstrated the Euclidean Postulate. And this may be the first demonstration physico-geometric.

I pass over to the second. Let there be a semicircle, of which the center is *D*, and diameter *AC*. If then (fig. 17) any point *B* is assumed in its circumference, to which *AB*, *CB* joined are ascertained to contain a right angle, this single case will be sufficient (as I have demonstrated in P. XVIII.) for establishing the hypothesis of right angle, and consequently (from the aforesaid P. XIII.) for demonstrating that well known Postulate.

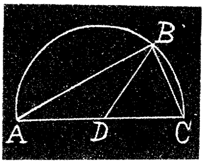

Fig. 17.

There remains the third demonstration physico-geometric, which I think the most efficacious, and most simple of all, inasmuch as it rests upon an accessible, most easy, and most convenient experiment.

For if in a circle, whose center is *D*, are fitted (fig. 22) three straight lines *CB*, *BL*, *LA*, each equal to the radius *DC*, and it is ascertained that the join *AC* goes through the center *D*, this will be sufficient for demonstrating the assertion.

For, *DB*, *DL* being joined, we will have three triangles, which (from Eu. I. 8 and 5) not only will be equiangular to one another, but also singly for themselves. Therefore since the three angles together at the point *D*, indeed *ADL*, *LDB*, *BDC* are equal (by Eu. I. 13) to two right angles; also the three angles together of each of these triangles will be equal to two right angles, as suppose of the triangle *BDC*. Wherefore (from P. XV.) will be established hence the hypothesis of right angle; and consequently (from the already used P. XIII.) that Postulate will be demonstrated.

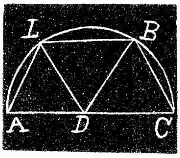

Fig. 22.

But if, before all attempt whether at demonstration or at graphic exhibition, one wishes to compare *inter se* those two postulates, I grant indeed the Euclidean may appear more obscure or even liable to objection. But after the graphic exhibition, which I reserve for Scholium IV. following, it will appear *vice versa* that the Euclidean Postulate indeed can retain the dignity and name of postulate, but the other ought rather to be reckoned among the theorems.

But here I must explain (as a little above I have promised I was about to do) the manifest disparity in this relation between the circular line and the straight line. Now the disparity arises from this; that a line is called straight in reference to itself; but is called circular, as suppose (fig. 23) *MDHNM*, not in reference to itself, but to something else, forsooth to a certain other point *A* existing in the same plane with it, which is its center.

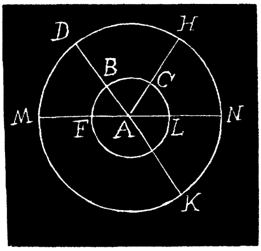

Fig. 23.

The consequence therefore is, as is demonstrated most excellently by Clavius, that the line *FBCL* existing in the same plane with it, and whose points are all equidistant from the aforesaid *MDHNM*, is also itself circular, truly equidistant in all its points from the common center *A*. That in fact *BD*, which is the continuation in a straight of *AB*, is the measure of the distance of that point *B* from this circle *MDHNM* follows from this; because (from Eu. III. 7, which is independent of the postulate here in controversy) this is the smallest of all, which can fall from this point upon this circumference. The same holds of the remaining *CH*, *LN*, *FM*.

Since therefore also the wholes *AM*, *AD*, *AH* are equal as radii from the center *A* to the line assumed circular *MDHNM*; and also the sections *FM*, *BD*, *CH*, *LN* are equal, which obviously are the measure of the equal distance of all points of that line *FBCLF* from this line presumed circular *MDHNM*; the consequence plainly is, that equal likewise are the remainders *AF*, *AB*, *AC*, *AL*, and therefore also this line is a circle with the same center *A*.

But now likewise, for demonstrating, that the line *DC* (fig. 21) generated through such a motion by the point *D* is a straight line will the equidistance of all its points from the underlying straight *BA* be sufficient? In no way.

For a line is called straight absolutely in reference to itself, or in itself, doubtless as *lying ex aequo between its points*, and especially end points, so that these remaining unmoved it cannot be revolved into occupying a new place. Unless this state in some way be demonstrated of this *DC* it will never be certain that this is a straight line, whatever relation finally is supposed or demonstrated of all its points to the underlying straight *AB* in the same plane; but especially we must not say analogically that no other line in this plane will be straight which in all its points is not equidistant from this line *AB* supposed straight.

Nor finally do I wish this dictum of mine so taken, as if I think it cannot be demonstrated, that the line thus generated is itself a straight line, except after truth demonstrated of the controverted postulate; since rather I myself will demonstrate it toward the end of this book, for confirming the like postulate itself.

[To be Continued.]